AN EXAMPLE CONCERNING UNIONS OF TWO STARSHAPED SETS IN THE PLANE

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ABSTRACT

Let S be a closed, simply connected subset of the plane, J a line segment (or a one-pointed set), $J \subseteq S$. If for every three points of S there is a point of J seeing at least two of these points via S, then S is a union of two starshaped sets. If $J \nsubseteq S$ or if S is not simply connected, the result fails.

Let S be a subset of R^n. For points x, y in S, we say x *sees* y *via* S if and only if the corresponding segment $[x, y]$ lies in S. Further, the set S is said to be *starshaped* if and only if there is some point p in S such that, for every x in S, p sees x via S. Valentine [1, Prob. 6.6, p. 178] has made the following conjecture: For S a closed set in R^n and J a line segment, if for each triple of points in S there is a point on J seeing at least two of these points via S, then S is a union of two starshaped sets. With stronger hypothesis, Valentine's conjecture is true in R^2, and it is not hard to prove the following result.

THEOREM. *Let S be a closed, simply connected subset of the plane, J a line segment (or a one-pointed set), $J \subseteq S$. If for every three points of S there is a point of J seeing at least two of these points via S, then S is a union of two starshaped sets. Further, if S contains no isolated point, we may require that the two starshaped sets be starshaped with respect to a point on J.*

PROOF. For each point x in S, let J_x denote the set of all points of J which x sees via S. Clearly each J_x is either empty, a singleton point, or a line segment. In case some $J_x = \varnothing$, then by Helly's theorem, $\bigcap \{J_y \colon y \text{ in } S \sim \{x\}\} \neq \varnothing$, and S will consist of the starshaped set $S \sim \{x\}$ and the isolated point $\{x\}$.

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Otherwise, let $J = [a, b]$ and $J_x = [a_x, b_x]$, $a \leqq a_x \leqq b_x \leqq b$ for every x in S. Define $a_0 = \sup\{a_x : x \in S\}$, $b_0 = \inf\{b_x : x \in S\}$. If $a_0 \leqq b_0$, then S is starshaped relative to both a_0 and b_0. If $b_0 < a_0$, then it is easy to show that every point of S sees either b_0 or a_0 via S.

However, in case $J \nsubseteq S$ or S is not simply connected, the result fails, as the following example illustrates.

Example. Let S denote the outer boundary and interior of Figure 1, J the line segment determined by a and b so that $J \cap S$ has two components. The only points seeing neither a nor b are in the shaded region, R, and for r in R, x in S, if x, r see no common point of $J \cap S$, then x sees b via A. Hence for every three points of S, some point of $J \cap S$ sees at least two of these points via S. However, S is not expressible as a union of two starshaped sets: there are no two points p, q of S such that each of the points u, v, w, y, z in Figure 1 sees one of p, q via S.

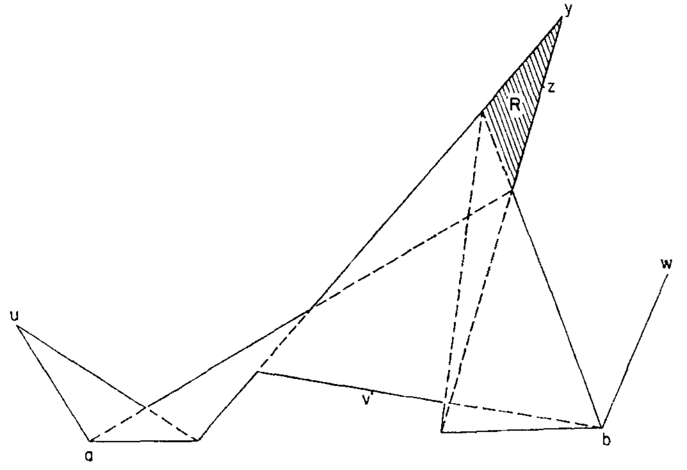

Fig. 1

Identical statements hold for the set $S \cup [a, b]$, which is not simply connected. Therefore, the strong hypothesis of the theorem is essential.

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The author wishes to thank the referee for his helpful suggestions, and for the following observation. The theorem remains true for closed sets in an arbitrary

linear topological space if we replace the condition that S be simply connected by the following assumption: The interior of every triangle having an edge on J, and the other two edges in S, is a subset of S.

REFERENCE

1. F. A. Valentine, *Convex Sets*, McGraw-Hill, New York, 1964.

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